AFOSR 2048

TR AE 6201

LINEARIZED DISSOCIATING GAS FLOW PAST SLENDER BODIES

by

T. Y. Li and K. C. Wang

Contract No. AF 49(638)-977

Mechanics Division
Air Force Office of Scientific Research
ARDC
Washington, D. C.

Department of Aeronautical Engineering and Astronautics
Rensselaer Polytechnic Institute
Troy, New York

January, 1962

TABLE OF CONTENTS

SUMMARY

In this report an inviscid, compressible, linearized dissociating gas flow over slender bodies of general cross section is considered. Essentially it is an extension of the conventional slender body theory include the dissociation effects. Both supersonic and subsonic cases are examined. To our present approximation the lateral forces and moments are found the same as those for the conventional case, but there exist additional non-equilibrium terms in the drag expression. If the body is pointed at both ends, the non-equilibrium term is formally the same for both the supersonic and the subsonic cases except that the sign in the respective cases is just the opposite. Physical aspects of the nonequilibrium drag problem are briefly considered. These considerations show that in the subsonic case the nonequilibrium term represents a drag and in the supersonic case it is a thrust. Finally, the supersonic drag of slender bodies of revolution at small angle of attack has been calculated, including dissociation effects.

Notation

a	Velocity of sound
C	Contour of the cross section in plane normal to x-axis (see below)
c	Drag coefficient
D	Drag
F^{-1}	Inverse Fourier transform
I_n, K_n	Modified Bessel function
$k = a_c/a_f{}^1$	Ratio of the equilibrium speed of sound to the frozen speed of sound
ℓ	Length of the body
L^{-1}	Inverse Laplace transform
M_f	Frozen Mach number
n	Normal to the body contour in y-z plane
p	Pressure
q	Velocity
r	Radial coordinate
R	Body radius
t	Thickness of the body
$\mathcal{s}$	Distance along C
s	Laplace transform variable
S	Cross section area
u,v,w	Velocity component in x, r, direction
x,y,z	Rectangular coordinates, x-axis being in the free stream direction

U	Free stream velocity
W	Complex potential
α	defined in Eqs. (3), (9)
α_o	Angle of attack
$\left.\begin{matrix}\beta_1\\ \beta_2\end{matrix}\right\}$	Defined in Eq. (3a)
γ	Euler's constant
ζ	Complex variable
θ	Angular coordinate
ρ	Mass density
$\tilde{\tau}$	Effective relaxation time, ref. (1)
ϕ	Perturbed potential
Φ	Fourier transform of
ψ	Stream function
ω	Fourier transform variable

Subscript

e	Equilibrium state
f	Frozen state
∞	Undisturbed state
r	Real part

Superscript

Bar	Laplace transform
Prime	Perturbed quantities

I. INTRODUCTION

The linearized inviscid compressible dissociating flows of a diatomic gas are irrotational and satisfy a generalized wave equation of third order in terms of the velocity potential ϕ . This equation was first derived by Moore[1] and has the same form as the equations for the propagation of elastic waves in a material subject to the relaxation of stresses, as obtained by Morrison[2].

By means of this linearized equation, several authors[1,3,4,5] have discussed some two-dimensional flow problems. Except for the case of a flow past a wavy wall[3], it was difficult[1,4,5] to determine the flow field. Therefore, in most of the recent results, only the conditions on the body surface were evaluated. For the study of an entire flow field, Moore and Gibson[1] further simplified the generalized wave equation to a variant of the telegraph equation for which solution may be obtained in a closed form.

The present report deals with linearized dissociating flow past slender pointed bodies of general cross-section. Essentially it is an extension of the earlier works of Ward and Adams and Sears[6,7], to include the dissociation effects. Both the supersonic and subsonic cases have been considered. The Laplace transform method is used for the supersonic case, the Fourier transform methods for the subsonic case. It will be seen that the trans-formed equations (transformed with respect to x) are identical in form as those obtained by Ward and Adams and Sears[6,7]. Then by the usual slender body approximations, the problem can be solved without any additional simplifications.

Non-equilibrium processes lead to irreversible increase in entropy along streamlines in the flow. However, this increase of entropy is of second order of magnitude,[8] consequently, to our linear approximation, the flow is still isentropic. Kusukawa and Li[8] have shown that the nonequilibrium drag due to the increase of entropy is of the third order in perturbation quantities. Such a nonequilibrium entropy drag will not be considered in the present paper.

We remark also that though we shall always refer our discussion to dissociation in this report, actually all the results apply equally to the case of vibrational relaxation because the linearized equation for these two cases is of the same form.

Development of the present theory was completed in late spring of 1961. Essentials of this work have been recorded previously in Wang's thesis[11] which was carried out under Li's direction. Independently, Clarke[12] studied the present problem and obtained results similar to the present theory. However, in Refs. (11) and (12) the authors held different opinions on the matter of the subsonic nonequilibrium drag on a doubly pointed slender body. Clarke concluded that in such a case the subsonic nonequilibrium drag is positive. Wang contended, on the other hand, that the subsonic nonequilibrium drag expression can take either positive or negative sign depending on the body shape (see Appendix of the present paper, also see Ref. 11). To be sure, the subsonic nonequilibrium drag problem was studied by Kusukawa and Li[8], using a thermodynamic approach. These recent calculations would support Clarke's conclusion. Publication of the present paper is undertaken at this time since it would complement the research results of Refs. 8 and 12. It is hoped that a quantitative analysis of the nonequilibrium drag effects will be made available shortly.

II. SOLUTIONS OF THE EQUATION

Consider a cylindrical coordinate system as shown in Fig. I, the free stream Velocity U being aligned with the x-axis, while the pointed nose being located at the origin. The body is slender, subject to the same restrictions as in the conventional slender body theory[6,7]. Its cross section may not be circular. The medium is taken to be a diatomic gas such as pure oxygen or nitrogen which forms a binary mixture when partially dissociated. In a linear theory the flow is irrotational, a perturbation potential ϕ may be defined:

$$\bar{q}' = \nabla\phi$$

where $\bar{q}'$ is the perturbation velocity. For steady flows, satisfies the following equation[1]

$$\frac{\check{\tau}\,\bar{u}_\infty}{k^2}\left[(1-M_f^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right] + \left[\left(1-\frac{M_f^2}{k^2}\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right] \qquad (1)$$

where

$$M_f = \frac{\bar{u}_\infty}{a_{f_\infty}}\,, \qquad k = \frac{a_{e_\infty}}{a_{f_\infty}} < 1$$

$\check{\tau}$ denotes the relaxation time, M_f the frozen Mach number, a_{e_∞} and a_{f_∞} are respectively the equilibrium speed of sound and the frozen speed of sound. It can be shown that a_{f_∞} is greater than a_{e_∞} so that, generally, k is less than unity.

For the ideal cases of equilibrium and frozen flows, there is only one speed of sound, a_{e_r} and a_{f_r} respectively. Eq. (1) consists of two parts (in two separate brackets), one for frozen flow, the other for equilibrium flow. When the relaxation $\hat{\tau}$ is long, the frozen part dominates; in the opposite case, the equilibrium part will govern the motion. In the following, we shall refer to flows as being "supersonic" or "subsonic" where both M_f and $\frac{M_f}{k}$ are greater or less than unity.

(a) Supersonic case

For the supersonic case, in front of the frozen Mach line from the nose the disturbances and the perturbation potential vanish. In the disturbed flow region, Eq. (1) can be written as

$$\frac{\hat{\tau}\,\bar{u}_r}{k^2}\left[(M_f^2-1)\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial r^2} - \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right]$$
$$+\left[\left(\frac{M_f^2}{k^2}-1\right)\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial r^2} - \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right] \qquad (1\text{-a})$$

Eq. (1-a) can be solved by the Laplace transform method. With the notation $s = \frac{\partial}{\partial x}$, the operational form of Eq. (1-a) is

$$\frac{\partial^2\bar{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\phi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\bar{\phi}}{\partial\theta^2} - \lambda^2\bar{\phi} = 0 \qquad (2)$$

where

$$\bar{\phi} = \int_0^\infty e^{-sx}\,\phi\,dx$$

and we have defined here

$$\alpha^2 = s^2(M_f^2 - 1)\left[\frac{s+\beta_1}{s+\beta_2}\right]$$

$$\beta_1 = \frac{k^2}{\tilde{\tau}\bar{u}_\infty}\ \frac{\frac{M_f^2}{k^2} - 1}{M_f^2 - 1} \tag{3}$$

$$\beta_2 = \frac{k^2}{\tilde{\tau}\bar{u}_\infty}$$

Since K $<$ 1, hence $\beta_1 > \beta_2$, α may be positive or negative, but always real. It is interesting to see that the transformed equation (2) assumes a form identical to that for the conventional case of non-dissociating flow. The difference appears only in α^2. For purely frozen or equilibrium flows Eq. (3) reduces to

$$\alpha^2 = s^2(M_f^2 - 1)$$

or

$$\alpha^2 = s^2(M_e^2 - 1)$$

The solution of Eq. (2) may be readily obtained, for example, by the separation of variables method in terms of modified Bessel functions $K_n(\alpha\lambda)$ and $I_n(\alpha\lambda)$. To have an outgoing wave propagating downstream from the body, we take

$$\bar{\phi} = \sum_{n=0}^{\infty} K_n(|\alpha|\lambda)\left[C_n(s)\cos n\theta + D_n(s)\sin n\theta\right]$$

where the coefficients C_n and D_n are to be determined presently.

Now we introduce the slender body approximation. For slender bodies $\frac{r}{\ell} << 1$, where ℓ is the length of the body. Hence to describe the flow on and near the body, we retain only the leading term in the series expansion of K_n, i.e.,

$$K_0(|\alpha| r) \cong \left(\log \frac{|\alpha| r}{2} + \gamma\right)$$

$$K_n(|\alpha| r) \cong \frac{2^{n-1}(n-1)!}{(|\alpha| r)^n} \qquad n = 1, 2, \ldots$$

where $\gamma = .5772\ldots$ is the Euler's constant. Thus

$$\bar{\phi} = -C_0(s)\left[\log \frac{|\alpha| r}{2} + \gamma\right] + \sum_{n=1}^{\infty} 2^{n-1}(n-1)!\left[\frac{C_n(s)\cos n\theta}{(|\alpha| r)^n} + \frac{D_n(s)\sin n\theta}{(|\alpha| r)^n}\right] \quad (4)$$

By inverse transform

$$\phi = a_0(x)\log r + b_0(x) + \sum_{n=1}^{\infty}\left[\frac{A_n(x)\cos n\theta}{r^n} + \frac{B_n(x)\sin n\theta}{r^n}\right] \quad (5)$$

where

$$\begin{aligned} a_0(x) &= \mathcal{L}^{-1}\left[-C_0(s)\right] \\ b_0(x) &= \mathcal{L}^{-1}\left[-C_0(s)\left(\log \frac{|\alpha|}{2} + \gamma\right)\right] \\ A_n(x) &= \mathcal{L}^{-1}\left[2^{n-1}(n-1)!\,\frac{C_n(s)}{|\alpha|^n}\right] \\ B_n(x) &= \mathcal{L}^{-1}\left[2^{n-1}(n-1)!\,\frac{D_n(s)}{|\alpha|^n}\right] \end{aligned} \quad (5a)$$

Introducing $\zeta = y + iz$ and a complex potential $W(\zeta, x)$, we have

$$W(\zeta, x) = \phi + i\psi = a_0(x)\log\zeta + b_0(x) + \sum_{n=1} \frac{a_n(x)}{\zeta^n} \quad (6)$$

where

$$a_n(x) = A_n(x) + i\,B_n(x) \tag{6a}$$

$a_0(x), a_n(x)$ will be determined by the boundary conditions. In particular, $a_0(x)$ can be determined in exactly the same way as in Ref. (6), so that

$$a_0(x) = \frac{\bar{u}_\infty\, S'(x)}{2\pi} \tag{7a}$$

The term $b_o(x)$ can be evaluated in terms of $a_0(x)$. From Eq. (3), we have

$$\log\frac{|k|}{2} = S\,\frac{\log S}{S} + \log\frac{\sqrt{M_f^2-1}}{2} + \frac{1}{2}\log\left[\frac{S+\beta_1}{S+\beta_2}\right]$$

The inversion of these terms are found[9] to be

$$\mathcal{L}^{-1}(S) = \delta'(x)$$

$$\mathcal{L}^{-1}\left[\frac{\log S}{S}\right] = -(\log x + \gamma)$$

$$\mathcal{L}^{-1}\left[\log\left(\frac{S+\beta_1}{S+\beta_2}\right)\right] = \frac{e^{-\beta_2 x} - e^{-\beta_1 x}}{x}$$

where $\delta(x)$ is the Dirac Delta function. Then by convolution, we obtain from Eq. (5a)

$$b_0(x) = a_0(x)\log\frac{\sqrt{M_f^2-1}}{2} - \int_0^x a_0'(\xi)\log(x-\xi)\,d\xi + \frac{1}{2}\int_0^x a_0(\xi)\,\frac{e^{-\beta_2(x-\xi)} - e^{-\beta_1(x-\xi)}}{x-\xi}\,d\xi \tag{7b}$$

The term involving $a(0)$ has been omitted because $a_0(0) \sim S'(0) = 0$ for pointed nosed bodies. The first two terms are same as those given by Ward[6] except for the appearance of the frozen Mach number in the present result. The last term gives the non-equilibrium effects. Any calculation not involving $b_o(x)$ will be free from the effects due to non-equilibrium dissociation, because it is the only place the non-equilibrium effects enter.

(b) Subsonic case

For the subsonic case, all disturbances die out at a large distance from the body, certainly they vanish at $x = \pm\infty$. Eq. (1) can be solved by the Fourier transform method. If Φ denotes the Fourier transform of ϕ with respect to x, i.e.,

$$\Phi = \int_{-\infty}^{\infty} \phi(x, r, \theta)\, e^{i\omega x}\, dx = F[\phi]$$

then the transformed form of Eq. (1) becomes

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} - \alpha^2 \Phi = 0 \qquad (8)$$

where

$$\alpha^2 = \omega^2 (1 - M_f^2)\left[\frac{\beta_1 - i\omega}{\beta_2 - i\omega}\right] \qquad (9)$$

with β_1 and β_2 similarily defined as in Eq. (3). Eq. (8) differs from that considered by Adams and Sears[7] only in , which now is a complex function of ω. This equation can be solved, under the restriction $\alpha_r > 0$, in exactly the same way as in the supersonic case. Thus we obtain

$$\Phi = \sum K_n(\alpha r)\left[C_n(\omega)\cos n\theta + D_n(\omega)\sin n\theta\right]$$

The modified Bessel functions with complex argument have similar asymptotic forms to those with real argument. For slender bodies we retain only the first term in the expansion series of K_n , the solution becomes formally identical with that for supersonic flow given in Eq. (4).

$$\Phi(\omega, r, \theta) = -C_0(\omega)\left[\log\frac{\alpha r}{2} + \gamma\right] + \sum_{n=1}^{\infty} 2^{n-1}(n-1)!\left[\frac{C_n(\omega)\cos n\theta}{\alpha^n r^n} + \frac{D_n(\omega)\sin n\theta}{\alpha^n r^n}\right] \quad (10)$$

Making inverse transform;

$$\phi(x, r, \theta) = a_0(x)\log r + b_0(x) + \sum_{n=1}^{\infty}\left[\frac{A_n(x)\cos n\theta}{r^n} + \frac{B_n(x)\sin n\theta}{r^n}\right] \quad (11)$$

where

$$a_0(x) = F^{-1}\left[-C_0(\omega)\right]$$

$$b_0(x) = F^{-1}\left[C_0(\omega)\left(\log\frac{\alpha}{2} + \gamma\right)\right]$$

$$A_n(x) = F^{-1}\left[2^{n-1}(n-1)!\,\frac{C_n(\omega)}{\alpha^n}\right] \quad (11a)$$

$$B_n(x) = F^{-1}\left[2^{n-1}(n-1)!\,\frac{D_n(\omega)}{\alpha^n}\right]$$

The inverse Fourier transform of a function $f(\omega)$ is defined by

$$F^{-1}\left[f(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega x}\, dx$$

$A_n(x)$ and $B_n(x)$ are real because ϕ is real. Again let $\zeta = re^{i\theta}$ and consider $\phi(x, r, \theta)$as the real part of a complex potential $W(\zeta, x)$ so that

$$W(\zeta, x) = a_0(x)\log\zeta + b_0(x) + \sum_{n=1}^{\infty}\frac{a_n(x)}{\zeta^n} \quad (12)$$

where

$$a_n(x) = A_n(x) + i\,B_n(x)$$

$a_0(x), a_n(x)$ must be determined as before by the boundary conditions, while $b_0(x)$ may be evaluated in terms of $a_0(x)$.

To evaluate $b_0(x)$, we rewrite Eq. (9)

$$\alpha^2 = \omega^2(1-M_f^2)\left[\frac{\omega^2+\beta_1^2}{\omega^2+\beta_2^2}\right]^{1/2} e^{i\theta} \quad ; \quad \theta = \tan^{-1}\frac{(\beta_1-\beta_2)\omega}{\beta_1\beta_2+\omega^2}$$

for $\alpha_r > 0$, we put

$$\alpha = |\omega|\sqrt{1-M_f^2}\left[\frac{\omega^2+\beta_1^2}{\omega^2+\beta_2^2}\right]^{1/4} e^{i\frac{\theta}{2}}$$

$$-\pi < \theta < \pi$$

so that

$$\log\frac{\alpha}{2} = \log|\omega| + \log\frac{\sqrt{1-M_f^2}}{2} + \frac{1}{4}\log\left[\frac{\omega^2+\beta_1^2}{\omega^2+\beta_2^2}\right] + i\frac{\theta}{2} \tag{13}$$

The inverse transform of these terms can be shown[9,10] to be

$$\begin{aligned}
F^{-1}\left[\log|\omega|\right] &= -\frac{\operatorname{Sgn} x}{2x} = -\frac{1}{2|x|} \\
F^{-1}\left[\log\frac{\sqrt{1-M_f^2}}{2}\right] &= \delta(x)\log\frac{\sqrt{1-M_f^2}}{2} \\
F^{-1}\left[\frac{1}{4}\log\frac{\omega^2+\beta_1^2}{\omega^2+\beta_2^2}\right] &= \frac{e^{-\beta_2|x|} - e^{-\beta_1|x|}}{4|x|} \\
F^{-1}\left[i\frac{\theta}{2}\right] &= \operatorname{Sgn} x\,\frac{e^{-\beta_2|x|} - e^{-\beta_1|x|}}{4|x|}
\end{aligned} \tag{14}$$

where $\delta(x)$ is the Dirac Delta function and $\operatorname{Sgn}(x)$ is defined by

$$\operatorname{sgn} x \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$$

then by convolution, we finally obtain

$$b_o(x) = a_o(x) \log \frac{\sqrt{1-M_f^2}}{2} - a_o(0)\frac{\log x}{2} - \frac{a_o(\ell)\log(\ell - x)}{2}$$
$$-\frac{1}{2}\int_0^x a_o'(\xi)\log(x-\xi)\,d\xi + \frac{1}{2}\int_x^\ell a_o'(\xi)\log(\xi - x)\,d\xi$$
$$+\frac{1}{2}\int_0^x a_o(\xi)\,\frac{e^{-\beta_2(x-\xi)} - e^{-\beta_1(x-\xi)}}{x-\xi}\,d\xi \qquad 0 \le x \le \ell \quad (15)$$

To save writing, we have heare used the condition that $a_o(x)$, $a_o'(x)$ vanish for $\ell < x$ and $x < 0$ as a direct result of Eq. (7a). Except for the appearance of the frozen Mach number the first five terms are the same as those given by Adams and Sears[7]. Again the last term of $b_o(x)$ gives the nonequilibrium effect. We observe that the nonequilibrium term has the same form for both the supersonic and subsonic flows.

From the classical hydrodynamical theory, we recognize immediately that the first term in Eqs. (6) and (12) represents a two-dimensional source while the summation terms represent a two-dimensional doublet ($n = 1$) and higher order signularities ($n > 1$). Therefore $W - b_o(x)$ represents an incompressible two-dimensional potential flow past a cylinder of general cross section.

The nonequilibrium effect appears only in the last term of $b_o(x)$. Since Eqs. (6) and (12) differ from the corresponding formulus in the conventional slender body theory only in $b_o(x)$, it follows that the nonequilibrium dissociation affects only the x-component of velocity u', not v' or w' in the radial or angular direction. An immediate consequence of this is that the nonequilibrium dissociation does not change the lateral forces and moments because they depend on the cross flow as will be shown later.

III. DRAG, LATERAL FORCES AND MOMENTS

In order to determine the aerodynamic forces on the body, some information about the order of magnitude of some pertinent quantities is needed. Following Ward's proof, we remark first that on the body the normal and tangential velocity components in a cross plane (perpendicular to x-axis) are both $O(t)$, consequently on and near the body

$$\frac{d\bar{W}}{d\xi} \sim O(t)$$

then Eqs. (6) and (12) indicate that

$$a_n(x) \sim O(t^{n+2}), \quad n = 0, 1, 2, \ldots$$

and

$$\begin{aligned} &\frac{\partial\phi}{\partial x}, \ \frac{\partial^2\phi}{\partial x^2} \sim O(t^2 \log t) \\ &\frac{\partial\phi}{\partial r}, \ \frac{1}{r}\frac{\partial\phi}{\partial\theta} \sim O(t) \\ &\frac{\partial^2\phi}{\partial r^2}, \ \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \sim O(1) \end{aligned} \tag{16}$$

In other words, the order of magnitude of these functions are not changed due to the nonequilibrium dissociation.

(a) Pressure relation

In deriving Eq. (1), we used the linearized Euler's equation

$$\frac{p - p_\infty}{\rho_\infty} = -\bar{u}_\infty\frac{\partial\phi}{\partial x}$$

which is just the linear approximation to the pressure. This pressure relation is usually good for two dimensional flow. For a slender body as we have just shown that $\frac{\partial\phi}{\partial x}$ and $\left(\frac{\partial\phi}{\partial r}\right)^2$, $\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2$ are of the same order of magnitude on and near the body, a better approximation is therefore

$$\frac{p - p_\infty}{\frac{1}{2}\rho_\infty \bar{u}_\infty^2} = -\left[\frac{2}{\bar{u}_\infty}\frac{\partial\phi}{\partial x} + \frac{1}{\bar{u}_\infty^2}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{\bar{u}_\infty^2}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right] \qquad (17)$$

which is the familiar quadratic approximation of the pressure for the slender body[13]. That the linearized equation (1), where the terms like $\left(\frac{\partial\phi}{\partial r}\right)^2$ etc. are neglected, in combination with the quadratic approximation of Eq. (17), where $\left(\frac{\partial\phi}{\partial r}\right)^2$ and $\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2$ are kept, gives the correct first approximation for the aerodynamic forces on the body is well known in the conventional slender body theory. We here merely demonstrated that the same pressure Eq. (17) should be used for the same reason, namely the terms on the right side are all of the same order of magnitude near the body.

(b) Density relation

In deriving Eq. (1), we also used the linearized continuity equation

$$\bar{u}_\infty \frac{\partial}{\partial x}\left(\frac{\rho - \rho_\infty}{\rho_\infty}\right) + \nabla^2\phi = 0$$

meanwhile Eqs. (5) and (11) indicate that in our present non-equilibrium case, ϕ also satisfies the Laplace's equation in the cross plane i.e.,

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

hence

$$\bar{u}_\infty \frac{\partial}{\partial x}\left(\frac{\rho-\rho_\infty}{\rho_\infty}\right) + \frac{\partial^2\phi}{\partial x^2} = 0$$

or

$$\frac{\rho}{\rho_\infty} \cong 1 - \frac{1}{\bar{u}_\infty}\frac{\partial\phi}{\partial x} \tag{18}$$

which is again the same form as used in the conventional case. Having thus shown that the same pressure and density relations as in the conventional case can be used for our present non-equilibrium case, we are now in a position to calculate the aerodynamic forces.

(c) Drag relation

Our solution ϕ given by Eqs. (5), (11), pressure Eq. (17), density Eq. (18) are of the identical forms as these for the conventional case, it follows that the expressions for the drag, lateral forces and moments deduced by Ward[6] from the general momentum theory apply equally to our present nonequilibrium flow, the drag is given by

$$\frac{D}{\frac{1}{2}\rho_\infty \bar{u}_\infty^2} = -\frac{2\pi}{\bar{u}_\infty^2} a_0(l)\, b_0(l) + \frac{4\pi}{u_\infty^2}\int_0^l a_0'(x)\, b_0(x)\, dx - \frac{1}{\bar{u}_\infty^2}\left[\oint \phi \frac{\partial\phi}{\partial n} ds\right]_{x=l} + C_{pB}S(l) \tag{19}$$

where C denotes the contour of the cross section of the body in a plane normal to the x-axis. $\frac{\partial\phi}{\partial n}$ denotes the derivative of the perturbation potential taken with respect to the outward normal to the body cross section contour (Fig. 2).

C_{pB} is the base pressure coefficient.

<u>For supersonic flow</u>, substitution $a_0(x)$, $b_0(x)$ from Eqs. (7a), (7b) gives under the assumption of pointed nose

$$\frac{D}{\frac{1}{2}\rho_\infty \bar{u}_\infty^2} = -\frac{1}{2\pi}\int_0^l\int_0^l S''(x)S''(\xi)\log|x-\xi|\,d\xi\,dx + \frac{S'(l)}{2\pi}\int_0^l S''(\xi)\log(l-\xi)\,d\xi$$

$$-\frac{1}{\bar{u}_\infty^2}\left(\oint_C \phi\frac{\partial\phi}{\partial n}ds\right)_{x=l} + C_{pB}\,S(l)$$

$$+\frac{1}{2\pi}\int_0^l\int_0^x S''(x)\,S'(\xi)\,\frac{e^{-\beta_2(x-\xi)}-e^{-\beta_1(x-\xi)}}{x-\xi}\,d\xi\,dx$$

$$-\frac{S'(l)}{4\pi}\int_0^l S'(\xi)\,\frac{e^{-\beta_2(l-\xi)}-e^{-\beta_1(l-\xi)}}{l-\xi}\,d\xi \quad . \tag{20}$$

where the first four terms are same as those obtained by Ward. The last two are purely due to the nonequilibrium dissociation. The third term contains the contributions due to the angle of attack and also the nonequilibrium dissociation because ϕ contains $b_0(x)$. This term will be evaluated later when a specific type of body like the body of revolution is to be considered.

It is interesting to notice that for pointed nose slender bodies, Eq. (20) does not depend on the Mach number, a fact which is well known in the conventional case. A consequence of this Mach number independence is that to the present approximation, a slender body experiences the same drag in an equilibrium or frozen dissociating gas flow as in the conventional gas flow, because the governing equations for these cases differ only in Mach number.

There are two cases for which Eq. (20) may be simplified. (a) when $S(\ell) = 0$ i.e., the body is pointed at both ends, and (b) when $S'(\ell) = 0$ and the generators of the cylinder are parallel to the free stream which makes $\frac{\partial \phi_0}{\partial n} \simeq \frac{dn}{ds} = 0$. For both of these cases, omitting the base pressure for case (b),

$$\frac{D}{\frac{1}{2}\rho_r u_r^2} = -\frac{1}{2\pi}\int_0^\ell\int_0^\ell S''(x)S'(\xi)\log|x-\xi|\,d\xi\,dx + \frac{1}{2\pi}\int_0^\ell\int_0^x S''(x)\,S'(\xi)\left[\frac{e^{-\beta_2(x-\xi)} - e^{-\beta_1(x-\xi)}}{x-\xi}\right]d\xi\,dx \tag{21}$$

Since $\beta_1 > \beta_2$, the quantity inside the bracket in the second integral is always positive. However the sign of $S''(x)$ and $S'(x)$ is, in general, dependent the body shape. The sign of the second integral therefore cannot be asserted, in other words, one cannot say whether it represents a drag or a thrust in general.

Though one can integrate the second integral in Eq. (21) by parts, and using the condition S'(0) = 0, to yield

$$\int_0^l \int_0^x S''(x)S''(\xi)\left[(\beta_1-\beta_2)(x-\xi) - \frac{\beta_1^2-\beta_2^2}{2\cdot 2!}(x-\xi)^2 + \frac{\beta_1^3-\beta_2^3}{3\cdot 3!}(x-\xi)^3 - \cdots\right] d\xi\, dx \qquad (21a)$$

or, by the definition of exponential integral, one can rewrite it as

$$\frac{1}{2} S'(l)^2 \log\frac{\beta_1}{\beta_2} + \int_0^l \int_0^x S''(x)S''(\xi)\left[Ei\{-\beta_2(x-\xi)\} - Ei\{-\beta_1(x-\xi)\}\right] d\xi\, dx$$

$$= \frac{1}{2} S'(l)^2 \log\frac{\beta_1}{\beta_2} + \frac{1}{2}\int_0^l \int_0^l S''(x)S''(\xi)\left[Ei\{-\beta_2|x-\xi|\} - Ei\{-\beta_1|x-\xi|\}\right] d\xi\, dx \qquad (21b)$$

it still does not appear to be possible that a definite conclusion on the sign may be drawn from any one of these alternative expressions in general. The situation is different for the first integral in Eq. (21) due to the different behavior of log $(x-\xi)$ and $\left\{Ei[-\beta_2(x-\xi)] - Ei[-\beta_2(x-\xi)]\right\}$. This first integral certainly gives a drag and is larger in magnitude. Some further discussion about the sign based on the mean value theorem is given in the Appendix.

In the case of a slender cone, R(x) = (constant)x. It follows then that

$$S'(x) = 2\pi R R' > 0$$

$$S''(x) = 2\pi R'^2 > 0$$

therefore $S''(x)\, S'(\xi) > 0$ and the second integral of Eq. (21) represents an increase in drag.

For subsonic flow, we consider the body to be pointed at both ends so we may assume the absence of the upstream influence of a wake. Eq. (19) thus becomes

$$\frac{D}{\frac{1}{2}\rho_f \bar{u}_f^2} = \frac{4\pi}{\bar{u}_f^2}\int_0^{\ell} a_0'(x)\, b_0(x)\, dx \qquad (19a)$$

on substituting $a_0(x)$, $b_0(x)$ from Eqs. (7a), (15), we have

$$\frac{D}{\frac{1}{2}\rho_f \bar{u}_f^2} = \frac{1}{2\pi}\int_0^{\ell}\int_0^{x} S''(x)\, S'(\xi)\left[\frac{e^{-\beta_2(x-\xi)} - e^{-\beta_1(x-\xi)}}{x-\xi}\right] d\xi\, dx \qquad (22)$$

the other terms either vanish by the assumption of pointed ends or cancel out each other. Eq. (22) says that there exist a non-equilibrium subsonic drag or thrust which in fact has the same form as its counterpart in supersonic flow given in Eq. (21) except that the sign is just reversed, because for subsonic flow, $\beta_1 < \beta_2$ the quantity inside the bracket in Eq. (22) is always negative. The possibility of a nonequilibrium thrust at subsonic speeds as indicated here deserves further attention. In fact, as pointed out in Introduction, Clarke[12] has obtained a subsonic nonequilibrium drag formula same as in Eq. (22) but he has concluded that the slender body would experience a positive non-equilibrium drag. To interpret the present result, we need to obtain some physical insight of the nonequilibrium drag problem. This is exactly what has been accomplished in Ref. 8. We shall discuss the physical aspects of the nonequilibrium drag probelm in Section IV.

(d) . Lateral forces and moments

The lateral forces F_z, F_y acting on the body can also be determined from the momentum considerations. However it is more convenient to write them in complex form

$$F = F_y + i F_z$$

so that the theory of complex function can be applied. Clearly the real part F_y in our present coordinate system represents the lift. The expression for F deduced by Ward depends only on the coefficient $a_1(x)$ and the base cross section, but not on the term $b_0(x)$. Consequently the lateral forces and their moments about the y and z-axis are the same for both supersonic and subsonic conventional gas flow.

For our present nonequilibrium flow, the nonequilibrium effects as discussed above, also enter only through the term $b_0(x)$. Since Ward's expression for the lateral forces and moments apply equally to our case, therefore we conclude the nonequilibrium dissociation does not affect the lateral forces and moments. To emphasize this point we may state that a slender body according to the present approximation experiences the same lateral forces and moments regardless of the flow being supersonic or subsonic, dissociating or non-dissociating, in equilibrium or frozen or in nonequilibrium. This conclusion is not surprising because the cross flow for all these cases is regarded as incompressible flow in the present approximation.

IV. PHYSICAL MECHANISM OF NONEQUILIBRIUM DRAG

Calculation in Section III show that a slender body would experience an additional drag (positive or negative, depending on the body shape) caused by chemical relaxation in a dissociating gas. The recent study of Kusukawa and Li provides an explanation of the physical mechanism of this nonequilibrium drag. They showed that the body drag can be computed as follows:

$$D = -\frac{1}{V_\infty}\iint d\sigma \int_{z=-\infty}^{\infty} p\, dV \tag{23}$$

where

$$V = \frac{1}{\rho} = \text{specific volume}$$

$$V_\infty = \frac{1}{\rho_\infty} = \text{Free stream sp vol}$$

$z=-\infty$ and $z=+\infty$ denote two points P and Q on a streamtube, upstream and downstream far from the body where all perturbations almost vanish and $\iint d\sigma$ denotes the integration with respect to all streamtubes, the streamtube cross section being $d\sigma$. From Eq. (23) it is seen that the sign of the drag is determined by the sign of the following integral along a streamtube:

$$-\int_{z=-\infty}^{z=+\infty} p\, dV = -\int_{z=-\infty}^{z=+\infty} (p_\infty + p')\, dV' \tag{23a}$$

where

$$p' = p - p_\infty$$
$$V' = V - V_\infty$$

The subscript ∞ denotes here free stream quantities and the primed quantities are perturbations*. Therefore if we can determine p' and V' along a streamtube we shall be able to compute the drag. Using the Lagrange's approach, Kusukawa and Li obtained a linear equation that provides the governing relation between the variation of p' and V' of a moving material element on a streamtube. In the case of subsonic flow past a slender body, the function V' between P and Q can be generally represented by a Fourier series:

$$V' = \sum_{n=1}^{\infty} a_n \sin 2\pi n \frac{t}{\vartheta^*} + \sum_{n=1}^{\infty} b_n \cos 2\pi n \frac{t}{\vartheta^*} \tag{24}$$

where t is the time variable, ϑ * denotes the time interval during which the material element moves from P to Q . The Fourier coefficients a_n and b_n must be determined by the dynamic and kinematic conditions appropriate to the problem and indeed they must be dependent on the body shape. In Ref. 8, it has been shown that the perturbation pressure p_n' that corresponds to a Fourier component

$$V_n' = a_n \sin 2\pi n \frac{t}{\vartheta^*} \tag{25}$$

* This should not be confused with the prime that denotes differentiation such as $S'(x)$ in Eq. (7a).

can be obtained as

$$p_n' = -a_n \sqrt{x^2+y^2}\, \sin\left(2\pi n \frac{t}{\vartheta^*} + \phi_n\right) \tag{26}$$

where
$$X = -\left[\left(\frac{2\pi n}{\vartheta^*}\tilde{\tau}\right)^2 + \frac{1}{k^2}\right] a_{f_r}^2 \Big/ \left[\left(\frac{2\pi n}{\vartheta^*}\tilde{\tau}\right)^2 + \frac{1}{k^4}\right] V_r^2$$

$$Y = -\frac{2\pi n}{\vartheta^*}\tilde{\tau}\left(\frac{1}{k^2}-1\right) a_{f_r}^2 \Big/ \left[\left(\frac{2\pi n}{\vartheta^*}\tilde{\tau}\right)^2 + \frac{1}{k^4}\right] V_r^2$$

$$\phi_n = \tan^{-1}\frac{Y}{X}$$

From Eqs. (25) and (26), we conclude that V_n' and p_n' are out of phase due to relaxation effects. In equilibrium and frozen flows, however, V_n' and p_n' are in phase and $\phi_n = 0$. The consequence of this phase shift due to relaxation effects is to cause a nonequilibrium drag as can be verified from application of these results in Eq. (23a). Indeed, in equilibrium and frozen flows, $\phi_n = 0$, $D = 0$, and in nonequilibrium flow, $\phi_n \neq 0$, Kusukawa and Li obtained

$$D = \rho_\infty \sum_{n=1}^{\infty} \iint (A_n^2 + B_n^2)\, d\sigma > 0 \tag{27}$$

where
$$A_n^2 = -\pi n a_n^2 Y > 0$$

$$B_n^2 = -\pi n b_n^2 Y > 0$$

Therefore, the phase shift ϕ_n for the functions V_n' and p_n' can be regarded the physical cause of the appearance of a nonequilibrium drag. This drag is positive by Eq. (27) because a_n and b_n, which are dependent on the body shape, have been assumed to be real quantities. This shows that for a class of bodies, for which Y' can be represented as in Eq. (24) with real a_n and b_n, we must expect the subsonic nonequilibrium drag to be positive. This also shows that the possibility of $D \sim A_n^2 + B_n^2$ exists only if $a_n^2 + b_n^2$ which implies that a_n and b_n must be complex quantities for such bodies. This possibility has not been seriously considered here since $\mathcal{V}'$ in Eq. (24) is a real physical quantity and generally a_n, b_n are expected to be real quantities. We remark therefore that the subsonic nonequilibrium drag on a doubly pointed slender body is positive. In this sense the integral in Eq. (22) should then take negative sign. Kusukawa and Li did not consider the supersonic flow case in detail. They pointed out that in such a case the drag would consist of supersonic wave drag and the nonequilibrium drag. The nonequilibrium effects would tend to provide damping of the disturbances propagating along Mach waves and would thus be expected to decrease the wave drag from the conventional value.

V. APPLICATION TO BODIES OF REVOLUTION

Since the lateral forces and moments are same as the conventional case, we shall consider here only the drag problem. Furthermore for subsonic flow, the body must be pointed at both ends so that our present theory may be applicable, and the drag has been completely determined in Eq. (22). In the following we therefore consider the supersonic flow only, and the body may have a flat base.

Our method of solution is first to obtain a complex potential $\bar{W} - \zeta_0(x)$ using our knowledge in the classical incompressible two dimensional flow. When a body of revolution is at a small angle of attack $-\alpha_0$ (Fig. III), we may consider the cross sections in a plane normal to the x-axis to be circles through they actually are ellipses with an error of $O(\alpha^2)$. It is well known that for a circular cylinder, the complex potential should be the superposition of a source and a doublet, i.e.

$$\bar{W} - \zeta_0(x) = \frac{\bar{u}\, S'(x)}{2\pi} \log(\zeta - \zeta_0) - \frac{\bar{u}\, S(x)}{\pi} \frac{\zeta_0'}{\zeta - \zeta_0} \qquad (28)$$

where is the center of the cross section, given by $\zeta_0 = -\alpha_0 x$ α_0 denotes the angle of attack. The coefficients were so chosen that the boundary conditions will be satisfied as shown below.

On the body $\zeta_s - \zeta_{s0} = R(x)\, e^{i\theta}$ substituting $S(x) = \pi R^2(x)$

$\zeta_{s0} = -\alpha_0 x$ in Eq. (28), we obtain

$$\bar{W}(x, R, \theta) = b_0(x) + \frac{\bar{u}\, S'(x)}{2\pi}\left[\log R(x) + i\theta\right] + \alpha_0 \bar{u}_\infty R(x) e^{-i\theta} \tag{29}$$

The real part in Eq. (29) is

$$\phi = b_0(x) + \frac{\bar{u}_\infty S'(x)}{2\pi} \log R(x) + \bar{u}_\infty \alpha R(x) \cos\theta \tag{30}$$

From the boundary condition, we have

$$\frac{\partial \phi}{\partial n} = \bar{u}_\infty \frac{dn}{dx}$$

for a body of revolution at angle of attack $-\alpha_0$

$$\frac{dn}{dx} = R'(x) - \alpha_0 \cos\theta$$

therefore, we obtain

$$\frac{\partial \phi}{\partial n} = \bar{u}_\infty \left[R'(x) - \alpha_0 \cos\theta \right] \tag{31}$$

This same relation can be obtained by differentiating Eq. (30). Thus, the coefficient in Eq. (28) have been chosen correctly. Furthermore, expanding Eq. (28), we obtain

$$W - b_0(x) = \frac{\bar{u}_\infty S'(x)}{2\pi} \log \zeta - \frac{\bar{u}_\infty \left(S'(x) \zeta_g + 2 S(x) \zeta_g' \right)}{2\pi \zeta}$$

which shows that

$$a_1(x) = -\bar{u}_v\left[\frac{s'(x)\,\tau_g + 2S(x)\,\tau_g'}{2\pi}\right]$$

while a_0 is the same as given by Eq. (7a).

Known ϕ and $\frac{\partial\phi}{\partial n}$, we can evaluate the contour integral $\left(\oint_C \phi \frac{\partial\phi}{\partial n} ds\right)_{x=l}$ in the drag expression Eq. (20)

$$\left[\oint \phi\frac{\partial\phi}{\partial n}ds\right]_{x=l} = \bar{u}_v\int_0^{2\pi}\left[b_0(l) + \frac{s'(l)}{2\pi}\log R(l) + \bar{u}_v R(l)\alpha_0\cos\theta\right]\left[R'(l) - \alpha_0\cos\theta\right]R(l)\,d\theta$$

$$= \bar{u}_v\left[b_0(l) + \frac{\bar{u}_v s'(l)}{2\pi}\log R(l)\right]2\pi R(l)R'(l) - \bar{u}_v\alpha_0^2\pi R^2(l)$$

$$= \bar{u}_v s'(l)\left[b_0(l) + \frac{\bar{u}_v s'(l)}{2\pi}\log R(l)\right] - \alpha_0^2 S(l)\,\bar{u}_v^2$$

Substituting $b_0(l)$ from Eq. (7b) in the above expression gives

$$\left[\oint \phi\frac{\partial\phi}{\partial n}ds\right]_{x=l} = \frac{\bar{u}_v^2}{2\pi}s'^2(l)\log\frac{\sqrt{M_v^2-1}}{2}R(l) - \frac{\bar{u}_v^2 s'(l)}{2\pi}\int_0^l s''(\xi)\log(l-\xi)\,d\xi$$

$$+ \frac{\bar{u}^2 s'(l)}{2\pi}\int_0^l s'(\xi)\,\frac{e^{-\beta_2(l-\xi)} - e^{-\beta_1(l-\xi)}}{l-\xi}\,d\xi - \alpha^2 S(l)\,\bar{u}_v^2 \qquad (32)$$

We may write the drag coefficient as follows:

$$C_D = \frac{D}{\frac{1}{2}\rho_v\bar{u}_v^2} = C_{D_0} + C_{D_1} + C_{D_{\text{non eq}}} \qquad (33)$$

where C_{D_0}, C_{D_1}, $C_{D_{\text{non eq}}}$ are respectively the drag coefficient at zero angle of attack, the induced drag coefficient and the non-equilibrium drag coefficient.

Substitution of Eq. (32) into Eq. (20) then yields

$$C_{D_0} = -\frac{1}{2\pi}\int_0^l\int_0^l s''(x)\, s''(\xi)\log|x-\xi|\,d\xi\,dx + \frac{s'(l)}{\pi}\int_0^l s''(\xi)\log(l-\xi)\,d\xi - \frac{s'^2(l)}{2\pi}\log\frac{\sqrt{M_1^2-1}}{2}\,R(l) \tag{34a}$$

$$C_{D_i} = \alpha^2 S(l) \tag{34b}$$

$$C_{D_{noneq}} = \frac{1}{2\pi}\int_0^l\int_0^x s''(x)\, s'(\xi)\,\frac{e^{-\beta_2(x-\xi)} - e^{-\beta_1(x-\xi)}}{x-\xi}\,d\xi\,dx - \frac{s(l)}{2\pi}\int_0^l s'(\xi)\,\frac{e^{-\beta_2(l-\xi)} - e^{-\beta_1(l-\xi)}}{l-\xi}\,d\xi \tag{34c}$$

We have thus demonstrated that the drag contributions due to the angle of attack and the nonequilibrium dissociation can be separated, as might have been expected in the present linearized theory.

VI Conclusions

In the present paper we have considered the inviscid compressible linearized dissociating gas flow over slender bodies of general cross section. Essentially it is an extension of the earlier works of Ward and Sears and Adams to include the dissociation effects. The transform methods have been used to obtain the solutions, both supersonic and subsonic case have been examined.

The effects of Mach number and the rate of dissociation enter only through the term $\phi_0(x)$. The pressure and density relations have been shown to take the same forms as in the conventional theory (c.f. Eqs. (17), (18)), consequently, the expressions for the lateral forces mements and drag deduced by Ward from general momentum theory apply equally to the present case.

For the equilibrium and frozen cases, the lateral forces moments and drag were found identical to those for the conventional case provided that the tail end of the body is either pointed or has a cylindrical form.

For the nonequilibrium case, the lateral forces and moments were found to be the same as in the conventional case, but there exist additional nonequilibrium terms in the drag expression. They may represent drag or thrust. If the body is pointed at both ends, the nonequilibrium term is the sam for both supersonic and subsonic flows except that the sign is just opposite. By considering the physical mechanism of nonequilibrium drag, we obtain that in the subsonic case, it represents a drag, in the supersonic case it is a thrust.

Application to bodies of revolution in the case of supersonic flow has been carried out in detail. The drag expression can be separated into three parts, (1) the drag at zero angle of attack, (2) the induced drag and (3) the nonequilibrium terms which may represent drag or thrust.

References

1. Gibson, W.E. and Moore, F.K., "Acoustic Propagation in a Diatomic Gas Subject to Thermal or Chemical Relaxation," Cornell Aeronautical Laboratory, AFOSR TN-58-1057.

2. Morrison, J.A., "Wave Propagation in Rods of Voigt Material and Visco Elastic Materials with Three Parameter Models," Quart. Appl. Math., Vol. 14, pp. 153-169 (1956-57).

3. Vincenti, W.G., "Non-equilibrium Flow Over a Wavy Wall." Dept. of Aero. Engrg., Stanford Univ., Report SUNAER No. 85, March 1959; or J. Fluid Mech., Vol. 6, Part 4, Nov. 1959.

4. Clarke, J.F., "The Linearized Flow of a Dissociating Gas," J. Fluid Mech., Vol. 7, Part 4, April 1960.

5. Li, T.Y., 'Recent Advances in Non-equilibrium Dissociating Gas Dynamics," RPI TR-AE-6008, AFOSR-49, Dec. 1960; also ARS Journal, Feb. 1961.

6. Ward, G.N., "Supersonic Flow Over Slender Pointed Bodies," Quart. J. Mech. and Appl. Math., Vol. II, Part I, 1949.
"Linearized Theory of Steady High Speed Flow," Cambridge University Press, 1955

7. Adams, M.C. and Sears, W.R., "Slender Body Theory Review and Extension," J. Aero. Sci., Feb. 1953.

8. Kusukawa, K. and Li, T.Y., "The Loss of Mechanical Energy in the Flow of a Dissociating Gas," Rensselaer Polytechnic Institute, TR AE 6111, Dec. 1961.

9. Erdelyi, Maynus, Oberhettinger, Tricomi, "Tables of Integral Transforms," Vol. I, McGraw Hill Book Co., 1954.

10. Lighthill, M.J., "Fourier Analysis and Generalized Functions," Cambridge Univ. Press, 1958.

11. Wang, K.C., "Linearized Dissociating Gas Flows - Part II," Rensselaer Polytechnic Institute, Ph.D. thesis, May 1961.

12. Clarke, J.F., "Relaxation Effects on the Flow Over Slender Bodies," the College of Aeronautics, Cranfield, England, CoA Note No. 115, March 1961.

13. Lighthill, M.J., "Supersonic Flow Past Bodies of Revolution," ARS R. and M. No. 8003, 1945.

Appendix

Estimation of Integral by Mean Value Theorem

Mean Value Theorem

Let (a) $\phi(x)$ and $\psi(x)$ be bounded and integrable in $[a,b]$ and (b) $\psi(x)$ keep the same sign throughout $[a,b]$, then

$$m\int_a^b \phi(x)\,dx < \int_a^b \psi(x)\phi(x)\,dx < M\int_a^b \phi(x)\,dx$$

where M and m are the upper and lower bounds of $\psi(x)$ in $[a,b]$, or we write

$$\int_a^b \phi(x)\,\psi(x)\,dx = \psi(\eta)\int_a^b \phi(x)\,dx \quad , \; a<\eta<b$$

For proof of this theorem, see, for example, Courant's Differential and Integral Calculus, Vol. I, p. 127, 2nd ed.

We now apply this theorem for the estimation of the integral inEq. (22). We know for the subsonic flow $\beta_1 < \beta_2$

$$f(x,\xi) = \frac{e^{-\beta_1(x-\xi)} - e^{-\beta_2(x-\xi)}}{x-\xi} > 0 \quad \text{for} \quad 0<\xi<x$$

hence

$$\frac{D}{\frac{1}{2}\rho_\infty \bar{u}_\infty^2} = \int_0^1\int_0^x s''(x)\,s'(\xi)\,f(x,\xi)\,dx\,d\xi$$

$$= \int_0^l S''(x)\, S'(\vartheta(x)) \int_0^x f(x,\xi)\, d\xi\, dx \qquad 0<\vartheta<x$$

$$= \int_0^l S''(x)\, S'(\vartheta(x)) \left[E_1(-\beta_2 x) - E_1(-\beta_1 x) + \log \frac{\beta_2}{\beta_1} \right] dx$$

Now since

$$\left[E_1(-\beta_2 x) - E_1(-\beta_1 x) + \log \frac{\beta_2}{\beta_1} \right] > 0$$

we may again apply the mean value theorem so that

$$\frac{D}{\frac{1}{2}\rho_\infty u_\infty^2} = S''(\vartheta')\, S''(\vartheta(\delta)) \int_0^l \left[E_1(-\beta_2 x) - E_1(-\beta_1 x) + \log \frac{\beta_2}{\beta_1} \right] dx \qquad \begin{matrix} 0<\vartheta<l \\ 0<\delta<l \end{matrix}$$

The last integral is positive, but both $S'(\delta)$ and $S'(\vartheta(\delta))$ may be positive and negative depending on the body's shape, consequently D may be positive or negative, i.e. drag or thrust.